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Photon-number statistics from the phase-averaged quadrature-field distribution: Theory and ultrafast measurement

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We prove that the photon-number distribution of an arbitrary single-mode optical state can be calculated directly from the phase-averaged quadrature amplitude distribution, measured using optical homodyne detection. We experimentally demonstrate the application of this result by measuring the ultrafast (subpicosecond), time-resolved photon-number statistics of a weak field from a pulsed diode laser. Also presented is a numerical calculation of the photon-number distribution of a quadrature-squeezed vacuum state.

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Recently, measurements of the probability distribution $P_\theta(x_\theta)$ of the quadrature-field amplitude x_θ of an optical field have been used for the experimental determination of optical quantum states [1]. The set of distributions $\{P_\theta(x_\theta)\}$ has been measured using optical homodyne detection (OHD), which is based on the interference of the signal field with a strong, coherent reference field (local oscillator), which has an adjustable phase with value θ [1,2]. Given the set of distributions for a sufficiently large number of θ values, the Wigner distribution of an arbitrary quantum state can be reconstructed using tomographic reconstruction [1,3]. The Wigner distribution of a quantum state is uniquely related to the density operator, and thus represents the complete experimental determination of the state of a given mode [4], which in turn yields the probability distributions for all possible observables, such as photon number and optical phase [5].

If the single-model state is known beforehand to have a uniform distribution of optical phase (phase-random state), the Wigner distribution can be calculated from a single quadrature amplitude distribution using an integral transform [6]. However, if the state does not have a uniform optical phase distribution, it is interesting and useful to ask whether the photon statistics of an arbitrary state can be obtained without measuring the full set $\{P_\theta(x_\theta)\}$ and using this to reconstruct the Wigner distribution. Here we present a method for doing this and experimentally demonstrate its application to a weak field from a pulsed diode laser. The demonstrated sampling time of the technique is 330 fs, which is over an order of magnitude faster than previously demonstrated techniques [7]. It should be straightforward to extend the sampling time to 10 fs.

Consider a case where a lack of precise phase control of the signal and local oscillator (LO) prevents measurements of the set of distributions $\{P_\theta(x)\theta)\}$ using OHD, and information on the measured optical state is thus incomplete. Here we address what information can be obtained from the phase-averaged quadrature-field amplitude distribution

$$\bar{P}(\xi)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\ P_\theta(x_\theta=\xi), \tag{1}$$

where ξ is used for the quadrature amplitude after θ averaging. These data would result from collecting a single quadrature-amplitude probability distribution using a LO field with a (uniformly) randomly modulated phase. We prove that the photon-number distribution $p(n)$ of any single-mode optical state can be calculated directly from the phase-averaged quadrature-amplitude distribution through a simple integral transform, without assuming any special form of the optical state. We present an analytical functional representation of this new transform.

The in-phase quadrature amplitude is defined in terms of the lowering operator $\hat{a}$ as $\hat{x}=(\hat{a}+\hat{a}^\dagger)/\sqrt{2}$ and has eigenstates $|x\rangle$. When an arbitrary signal field in state $|\Psi\rangle$ passes through an optical phase shifter (e.g., a block of glass) the resulting phase-shifted state is $\hat{U}(\theta)|\Psi\rangle$, where $\hat{U}(\theta)=e^{-i\theta\hat{n}}$, with $\hat{n}=\hat{a}^\dagger\hat{a}$ and θ is the optical phase shift [8,9]. When using a LO field that has been phase shifted by θ, optical homodyne detection measures the distribution of the phase-shifted quadrature amplitude $\hat{x}_\theta=(\hat{a}e^{i\theta}+\hat{a}^\dagger e^{-i\theta})/\sqrt{2}$, which is represented by the basis $|x_\theta\rangle=\hat{U}(\theta)|x\rangle$. For a signal field state described by the density operator $\hat{\rho}$, the measured quadrature distribution is

$$P_\theta(x_\theta)=\langle x_\theta|\hat{\rho}|x_\theta\rangle=\langle x|\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)|x\rangle. \tag{2}$$

This shows that phase shifting the LO field by θ corresponds to shifting the signal field by $-\theta$, so we will denote the phase-shifted density operator as $\hat{\rho}(-\theta)\equiv\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)$.

The phase-averaged quadrature probability distribution given in Eq. (1) can be rewritten as

$$\bar{P}(\xi)=\frac{1}{2\pi}\int_0^{2\pi}d\theta\langle x|\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)|x\rangle|_{x=\xi}=\langle x|\hat{\rho}_r|x\rangle|_{x=\xi}, \tag{3}$$

where we defined a density operator $\hat{\rho}_r$ for the phase-randomized version of the actual state $\hat{\rho}$,

$$\hat{\rho}_r=\frac{1}{2\pi}\int_0^{2\pi}d\theta\ \hat{\rho}(-\theta). \tag{4}$$

Next we prove the intuitively reasonable result that the photon-number distribution does not change when the optical phase of an arbitrary state is randomized [8]. The photon-number distribution of the phase-random state $\hat{\rho}_r$ is given by

$$p(n)=\langle n|\hat{\rho}_r|n\rangle=\frac{1}{2\pi}\int_0^{2\pi}d\theta\langle n|\hat{U}^\dagger(\theta)\hat{\rho}\hat{U}(\theta)|n\rangle. \quad (5)$$

Using

$$\hat{U}(\theta)|n\rangle=e^{-i\theta\hat{n}}|n\rangle=|n\rangle e^{-i\theta n} \quad (6)$$

in Eq. (5), it is apparent that

$$p(n)=\langle n|\hat{\rho}_r|n\rangle=\langle n|\hat{\rho}|n\rangle. \quad (7)$$

Equation (3) shows that the phase-averaged quadrature distribution of an optical state is equivalent to the fixed-phase quadrature distribution of the phase-randomized state, and Eq. (7) shows that the photon-number distribution of an optical state does not change if the phase of that state is randomized. Therefore, we are left to derive the photon-number distribution of a phase-random state given its (single) quadrature probability distribution. The transformation between the quadrature distribution of a phase-random state and its resulting Wigner distribution $W(x,p)$ has previously been derived by Leonhardt and Jex and is given by [6]

$$W(r)=\frac{-1}{2\pi}\left\{\int_{+r}^{\infty}d\xi-\int_{-\infty}^{-r}d\xi\right\}\frac{\partial_\xi\bar{P}(\xi)}{\sqrt{\xi^2-r^2}}, \quad (8)$$

where $W(r)$ depends only on $r=\sqrt{x^2+p^2}$ since the state is phase random, and ∂_ξ denotes a derivative.

The photon-number distribution $p(n)$ can be obtained from the Wigner distribution by integrating Laguerre polynomials $L_n(x)$ over the Wigner distribution [10]

$$p(n)=2(-1)^n\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}dp\, L_n[2(x^2+p^2)] \times e^{-(x^2+p^2)}W(x,p). \quad (9)$$

Transforming Eq. (9) into polar coordinates and using Eq. (8), one obtains the photon-number distribution directly from the phase-averaged quadrature probability distribution as

$$p(n)=-2(-1)^n\int_0^{\infty}r\,dr\,L_n(2r^2) \times e^{-r^2}\left\{\int_{+r}^{\infty}d\xi-\int_{-\infty}^{-r}d\xi\right\}\frac{\partial_\xi\bar{P}(\xi)}{\sqrt{\xi^2-r^2}}. \quad (10)$$

This can be rewritten as

$$p(n)=-2(-1)^n\int_{-\infty}^{\infty}d\xi\;\partial_\xi\bar{P}(\xi)Z_n(\xi), \quad (11)$$

where

$$Z_n(\xi)=\int_0^{\infty}r\,dr\frac{L_n(2r^2)e^{-r^2}}{\sqrt{\xi^2-r^2}}[\Theta(\xi-r)-\Theta(-\xi-r)], \quad (12)$$

and $\Theta(x)$ is the Heaviside step function. Integrating Eq. (11) by parts one obtains the simple form

$$p(n)=2(-1)^n\int_{-\infty}^{\infty}d\xi\;\bar{P}(\xi)\partial_\xi Z_n(\xi). \quad (13)$$

Equation (12) can be simplified using the properties of the step function and changing variables to $q=r^2/\xi^2$, and substituting the power-series representation of the Laguerre polynomial [11] into Eq. (12) yields

$$Z_n(\xi)=\frac{\xi}{2}\sum_{m=0}^{n}\frac{(-1)^m}{m!}\binom{n}{m}2^m\xi^{2m}\int_0^1 dq\frac{q^m e^{-q\xi^2}}{\sqrt{1-q}}$$
$$=\sum_{m=0}^{n}\frac{(-1)^m}{(2m+1)!!}\binom{n}{m}2^{2m+1}\xi^{2m+1} \times\Phi(m+1,m+\tfrac{3}{2};-\xi^2), \quad (14)$$

where $\Phi(\alpha,\beta;z)$ is a degenerate (confluent) hypergeometric function.

Taking the derivative with respect to ξ of Eq. (14) and using the properties of the hypergeometric function, and combining with Eq. (13), we obtain the final result

$$p(n)=\int_{-\infty}^{\infty}d\xi\;M_n(\xi)\bar{P}(\xi), \quad (15)$$

where the function $M_n(\xi)$ is given by

$$M_n(\xi)=\sum_{m=0}^{n}\frac{(-1)^{m+n}}{(2m+1)!!}\binom{n}{m}2^{2m+1}(2m+1)\xi^{2m} \times\Phi(m+1,m+\tfrac{1}{2};-\xi^2). \quad (16)$$

Thus, the photon-number distribution of an arbitrary optical state can be determined by averaging the function $M_n(\xi)$ over the phase-averaged quadrature probability distribution. This result is in the same spirit as recent work on sampling the density matrix using optical homodyne detection [12].

The function $M_n(\xi)$ can be calculated using the power-series representation for $\Phi(\alpha,\beta;z)$. As can be seen from Eq. (16) the function $M_n(\xi)$ is an even function in ξ. Figure 1 shows plots of the function $M_n(\xi)$ for n equal to 0 and 9. Once the set of functions $\{M_n(\xi)\}$ is calculated, the photon-number distribution can be easily and quickly calculated from the phase-averaged quadrature probability distribution measured using OHD.

The use of OHD in the detection and characterization of squeezed light is widespread [13]. In the case of quadrature-squeezed light the precise control of the phase difference

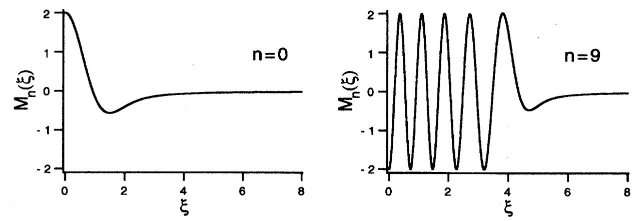

FIG. 1. Calculated value of the function $M_n(\xi)$ versus ξ for n equal to 0 and 9.

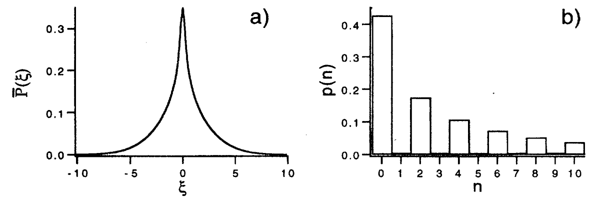

FIG. 2. (a) Theoretical phase-averaged quadrature probability distribution for a quadrature-squeezed vacuum state and (b) the resulting calculated photon-number distribution.

between the signal and the LO is necessary in order to observe high levels of squeezing. On the other hand, we have shown here that if only the photon-number distribution is sought, the difficulty of phase control and tomographic reconstruction can be eliminated. As a first example of the usefulness of this theory, we present a numerical calculation of the photon-number distribution of a quadrature-squeezed vacuum state calculated from the theoretical phase-averaged quadrature field distribution.

The quadrature probability distribution for a quadrature-squeezed vacuum state is given by [14]

$$P_\theta(\xi)=\frac{1}{\sqrt{\pi w_\theta^2}}\exp\left(\frac{-\xi^2}{w_\theta^2}\right), \tag{17}$$

where $w_\theta^2=\cosh(s)+\sin(2\theta)\sinh(s)$, s is the gain parameter, and the quantum efficiency of the detection system is assumed to equal unity. By numerically integrating Eq. (17) we calculated the phase-averaged quadrature probability distribution. Figure 2(a) shows a plot of $\bar{P}(\xi)$ for $s=3.0$. By integrating $\bar{P}(\xi)$ with the set of functions $\{M_n(\xi)\}$, we then calculate the photon-number distribution of the squeezed state, which is shown in Fig. 2(b). Although all phase information on the squeezed state is lost by averaging over the LO phase, the photon-number distribution of the state and the even-odd oscillations in photon number characteristic of quadrature-squeezed light can be calculated.

A case in which the phase difference between the LO and signal cannot be controlled is that of a pulsed laser. Although a laser pulse has a well-defined phase on each pulse, the field builds up from a few, random spontaneous emission events, and thus has a random phase from pulse to pulse.

We have demonstrated the use of dc-balanced OHD [1] to measure the time-resolved photon-number statistics of a 5-ns pulsed field with a sampling time of 330 fs, set by the duration of the LO pulse. The ultrafast sampling allows us to measure the photon statistics of very broadband fields such as a diode laser below threshold [15]. We use an argon-laser-pumped Ti:sapphire laser in combination with a chirped-pulse regenerative amplifier to generate ultrashort, transform-limited LO pulses (330 fs) at a wavelength of 830 nm and a repetition rate of 4 kHz with approximately 10^6 photons per pulse. Our signal is from a Sharp LT015 laser diode with a wavelength of 830 nm and a pulse width of 5 ns. The laser-diode-current drive pulse is triggered by the LO through an adjustable electronic delay having 80-ps jitter, which sets the time-resolution limit in this experiment. Because the overall efficiency of the homodyne detection system ($\sim$65%) is less than 100%, we should say that the measured $p(n)$ corresponds to an effective field that has suffered detection losses.

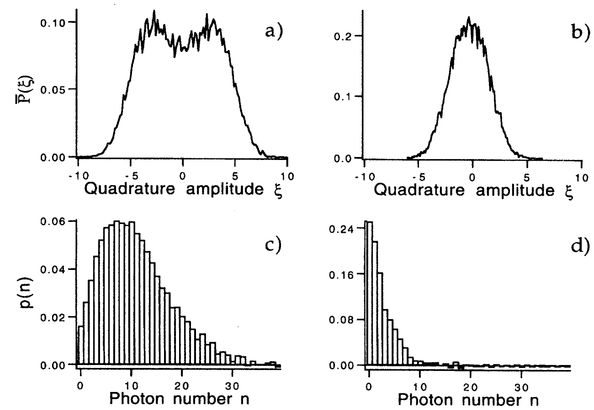

FIG. 3. Measured quadrature-amplitude distributions for (a) $t=4.0$ ns and (b) $t=6.0$ ns, and the resulting photon-number distributions for (c) $t=4.0$ ns and (d) $t=6.0$ ns, obtained from (a) and (b) by using Eq. (15).

Figure 3 shows examples of the measured quadrature probability distributions and the resulting photon-number probability distributions at two different times in the pulse, at [(a) and (c)] 4 ns and [(b) and (d)] 6 ns after the laser diode turns on. The distribution in Fig. 3(c) shows nearly Poisson statistics, as expected for a laser above threshold. Figure 3(d) shows a transition to thermal-like statistics as the laser drops below threshold near the end of the pulse.

Figure 4 shows the average photon number $\langle n\rangle$ during the 330-fs sampling time, obtained from quadrature distributions measured at 80 different time delays between the LO and the beginning of the signal pulse. Relaxation oscillations with a period of 300 ps are observed in the average photon number $\langle n\rangle$ as the laser first turns on.

In conclusion, we have derived a transformation between the photon-number distribution of an arbitrary optical state and the single phase-averaged quadrature-field amplitude distribution. This transformation is simply an integration of the quadrature distribution over a set of well-behaved, analytical functions $\{M_n(\xi)\}$. Since the phase-averaged quadrature distribution can be measured using OHD with a phase-

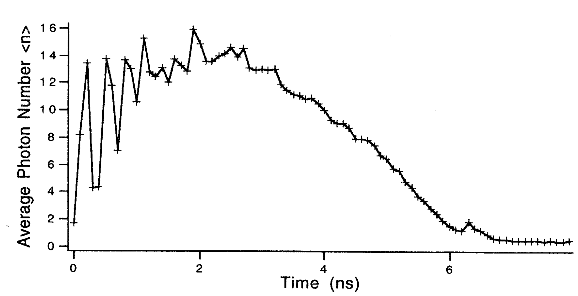

FIG. 4. Measured average photon number versus time delay between the LO and start of the signal pulse.

random LO, this theory is useful in all cases where phase control of the optical fields is difficult or impossible. The use of OHD using ultrashort phase-random LO pulses to measure time-resolved photon-number statistics offers promise for time resolving the photon statistics of weak, ultrafast optical sources such as molecular and semiconductor systems.

Note added in proof. Subsequent to the submission of this paper, a paper with similar theoretical results was presented at the 3rd Central-European Workshop on Quantum Optics: H. Paul, U. Leonhardt, and G. M. D'Ariano, Acta Phys. Slov. **45**, 261 (1995).

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